On Hausdorff dimension of oscillatory motions of the Sitnikov and the restricted planar circular three body problem

Part I: Conservative Newhouse Phenomena

PRELIMINARY VERSION

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0.1. Introduction.

S. Newhouse [N] showed that near every dissipative surface diffeomorphism with a homoclinic tangency there are open sets (nowadays called *Newhouse domains*) of maps with persistence homoclinic tangencies. Moreover, in these open sets there are residual subsets of maps with infinitely many attracting periodic orbits. Later C.Robinson [R] noticed that this result can be formulated in terms of generic one parameter unfolding of a homoclinic tangency.

P. Duarte [Du1], [Du2] showed that in area preserving case homoclinic tangencies also lead to similar phenomena, the role of sinks is played by elliptic points.[1] Here we prove a stronger one parameter version of the result: we can estimate the Hausdorff dimension of the hyperbolic sets and homoclinic classes that appear in the construction.

We would like to mention the following related results. S.Newhouse [N4] proved that in $\mathrm{Diff}^1(M^2, Leb)$ there is a residual subset of maps such that every homoclinic class for each of those maps has Hausdorff dimension 2. Moreover, Arnaud, Bonatti and Crovisier [BC], [ABC] improved that result and showed that in the space of C^1 symplectic maps the residual subset consists of the transitive maps that have only one homoclinic class (the whole manifold). Notice that this result can not be extended to higher smoothness.

T.Downarowicz and S.Newhouse [DN] proved also that there is a residual subset $\mathcal{R}$ of the space of C^r-diffeomorphisms of a compact two dimensional manifold M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has compact invariant topologically transitive sets of Hausdorff dimension two.

Initially our interest in the conservative Newhouse phenomena was motivated by the fact that it appears in the three body problem. Namely, let us try to understand the structure of the set of oscillatory motions (a planet approaches infinity always returning to a bounded domain) in the Sitnikov problem. It is a special case of the restricted three body problem where the two primaries with equal masses are moving in an elliptic

[1] From D.Turaev we learned recently that later P.Duarte also proved a one parameter version of the result (unpublished)

orbits of the two body problem, and the infinitesimal mass is moving on the straight line orthogonal to the plane of motion of the primaries which passes through the center of mass. The eccentricity of the orbits of primaries is a parameter. After some change of coordinates (McGehee transformation) the infinity can be considered as a degenerate saddle with smooth invariant manifolds that correspond to parabolic motions (the orbit tends to infinity with zero limit velocity). Stable and unstable manifolds coincide in the case of circular Sitnikov problem (parameter is equal to zero). Dankowicz and Holmes [DH] showed that for non-zero eccentricity invariant manifolds have a point of transverse intersection. This leads to the existence of homoclinic tangencies and appearance of all the phenomenon that can be encountered in the conservative homoclinic bifurcations, as described in the previous section. In particular, existence of hyperbolic sets of large Hausdorff dimension implies that *there is a sequence of open intervals in the space of parameters that accumulates to zero, and residual subsets of these intervals such that for corresponding parameters the set of oscillatory orbits in the Sitnikov problem has full Hausdorff dimension. Similar statement holds for the planar circular restricted three body problem.* The existence of transversal homoclinic points in the latter case was established in [LS], [X].

I would be extremely interesting to find out if Theorem 2 below can be strengthened by replacing the statement about the full Hausdorff dimension by a similar statement about positive Lebesgue measure of a homoclinic class. The question is related to Kolmogorov's Conjecture which claims that the set of oscillatory motions has zero measure.

This text is intended to become the part of joint paper with V.Kaloshin.

0.2. **Main results.**

Theorem 1. *Consider the family of area preserving Henon maps*

$$H_a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + a - y^2 \end{pmatrix}. \tag{1}$$

There is a (piecewise continuous) family of sets Λ_a such that the following properties hold.

1. The set Λ_a is a locally maximal hyperbolic set of the map H_a;

2. The set Λ_a contains a saddle fixed point of the map H_a;

3. The set Λ_a has an open and closed (in Λ_a) subset $\widetilde{\Lambda}_a$ such that the first return map for $\widetilde{\Lambda}_a$ is a two-component Smale horseshoe, and $\tau_{LR}(\widetilde{\Lambda}_a) \to \infty$ as $a \to -1$;

4. $dim_H \widetilde{\Lambda}_a \to 2$ as $a \to -1$.

Remark 1. *The proof will essentially use:*

• *the construction by Duarte* [Du1], [Du2] *who proved persistency of homoclinic tangencies for conservative maps near the identity, and*

• *results regarding the splitting of separatrices for Henon family,* [G1], [G2], [G3], [GS] *(see also* [Ch1] *and* [Ch2], *where some numerical results are described).*

Remark 2. *A similar statement holds also for any generic one parameter unfolding of an extremal periodic point (see* [Du1] *for a formal definition) as soon as the form of the splitting of separatrices can be established (see* [G1] *for the relevant results in that direction).*

Definition 1. *Let P be a hyperbolic saddle of a diffeomorphism f. A homoclinic class $H(P, f)$ is a closure of the union of all the transversal homoclinic points of P.*

It is known that $H(P, f)$ is a transitive invariant set of f, see [N1]. Moreover, consider all basic sets (locally maximal transitive hyperbolic sets) that contain saddle P. A homoclinic class $H(P, f)$ is a smallest closed invariant set that contains all of them.

Theorem 2. *Let $f_0 \in \mathrm{Diff}^\infty(M^2, Leb)$ have an orbit $\mathcal{O}$ of quadratic homoclinic tangencies associated to some hyperbolic fixed point P_0, and $\{f_\mu\}$ be a generic unfolding of f_0 in $\mathrm{Diff}^\infty(M^2, Leb)$. Then there is an open set $\mathcal{U} \subseteq \mathbb{R}^1$, $0 \in \overline{\mathcal{U}}$, such that the following holds:*

(1) *for every $\mu \in \mathcal{U}$ the map f_μ has a basic set that contains the unique fixed point P_μ near P_0 and exhibits persistent homoclinic tangencies;*

(2) *there is a dense subset $\mathcal{D} \subseteq \mathcal{U}$ such that for every $\mu \in \mathcal{D}$ there is some homoclinic tangency of the fixed point P_μ;*

(3) *there is a residual subset $\mathcal{R} \subseteq \mathcal{U}$ such that for every $\mu \in \mathcal{R}$*

(3.1) *the homoclinic class $H(P_\mu, f_\mu)$ is accumulated by f_μ's generic elliptic points,*

(3.2) $\dim_H H(P_\mu, f_\mu) = 2$,

(3.3) $\dim_H \{x \in M \,|\, P_\mu \in \omega(x) \cap \alpha(x)\} = 2$.

0.3. **Left-right thickness.**

Here we reproduce the definition of the left and right thickness from [Du2] and [Mo]. We will use these one-sided thicknesses instead of the standard definition. See [PT] for the usual definition of the thickness of a Cantor set.

Name *dynamically defined Cantor set* any pair (K, ψ) such that $K \subseteq \mathbb{R}$ is a Cantor set and $\psi : K \to K$ is a locally Lipschitz expanding map, topologically conjugated to some subshift of a finite type of a Bernoulli shift $\sigma : \{0, 1, \dots, p\}^{\mathbb{N}} \to \{0, 1, \dots, p\}^{\mathbb{N}}$. For the sake of simplicity, and because this is enough for our purpose, we will restrict ourselves to the case where ψ is conjugated to the full Bernoulli shift $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$. Also we will assume that a *Markov partition* $\mathcal{P} = \{K_0, K_1\}$ of (K, ψ) is given. In our case this means that the following properties are satisfied:

(1) $\mathcal{P}$ is a partition of K into disjoint union of two Cantor subsets, $K = K_0 \cup K_1$, $K_0 \cap K_1 = \emptyset$;

(2) the restriction of ψ to each K_i, $\psi|_{K_i} : K_i \to K$, is a strictly monotonous Lipschitz expanding homeomorphism.

For a general definition of Markov partition see [Mo], [PT].

Given a symbolic sequence $(a_0, \dots, a_{n-1}) \in \{0, 1\}^n$, denote

$$K(a_0, \dots, a_{n-1}) = \cap_{i=0}^{n-1} \psi^{-i}(K_{a_i}),$$

then the map $\psi^n : K(a_0, \dots, a_{n-1}) \to K$ is a Lipschitz expanding homeomorphism.

A bounded component of the complement $\mathbb{R} \backslash K$ is called a *gap* of K. For a dynamically defined Cantor set (K, ψ) the gaps are ordered in the following way. Denote by $\widehat{A}$ the convex hall of a subset $A \subseteq \mathbb{R}$. Then the interval $\widehat{K} \backslash (\widehat{K_0} \cup \widehat{K_1})$ is called a gap of order zero. A connected component of

$$\widehat{K} \backslash \cup_{(a_0, \dots, a_{n-1}) \in \{0,1\}^n} \widehat{K(a_0, \dots, a_{n-1})}$$

that is not a gap of order less than or equal to $n - 1$ is called a gap of order n. It is straightforward to check that every gap of K is a gap of some finite order, and also that, given a gap $U = (x, y)$ of order n, for every $0 \leq k \leq n$ the open interval bounded by $\psi^k(x)$ and $\psi^k(y)$ is a gap of order $n - k$.

Definition 2. *Given a gap U of K with order n, we denote by L_U, respectively R_U, the unique interval of the form $\widehat{K(a_0, \dots, a_{n-1})}$, with $(a_0, \dots, a_{n-1}) \in \{0, 1\}^n$, that is left, respectively right, adjacent to U. The greatest lower bounds*

$$\tau_L(K) = \inf \left\{ \frac{|L_U|}{|U|} : U \text{ is a gap of } K \right\}$$

$$\tau_R(K) = \inf \left\{ \frac{|R_U|}{|U|} : U \text{ is a gap of } K \right\}$$

are respectively called the left and right thickness of K. Similarly, the ratios

$$\tau_L(\mathcal{P}) = \frac{|L_{U_0}|}{|U_0|} \quad \text{and} \quad \tau_R(\mathcal{P}) = \frac{|R_{U_0}|}{|U_0|},$$

where U_0 is the unique gap of order zero, are called the left and the right thickness of the Markov partition $\mathcal{P}$.

Definition 3. *Given a Lipschitz expanding map $g : J \to \mathbb{R}$, defined on some subset $J \subset \mathbb{R}$, we define distortion of g on J in the following way:*

$$\mathrm{Dist}(g, J) = \sup_{x,y,z \in J} \log \left\{ \frac{|g(y) - g(x)|}{|g(z) - g(x)|} \frac{|z - x|}{|y - x|} \right\} \in [0, +\infty],$$

where the sup is taken over all $x, y, z \in J$ such that $z \neq x$ and $y \neq x$; due to injectivity of g this implies that $g(z) \neq g(x)$ and $g(y) \neq g(x)$.

Reversing the roles of y and z we see that the distortion is always greater than or equal to $\log 1 = 0$. If $\mathrm{Dist}(g, J) = c$, then for all $x, y, z \in J$ with $z \neq x$ and $y \neq x$ we have

$$e^{-c} \frac{|y - x|}{|z - x|} \leq \frac{|g(y) - g(x)|}{|g(z) - g(x)|} \leq e^{c} \frac{|y - x|}{|z - x|}.$$

Definition 4. *The distortion of a dynamically defined Cantor set (K, ψ) is defined as*

$$\mathrm{Dist}_\psi(K) = \sup \mathrm{Dist}(\psi^n, K(a_0, \dots, a_{n-1}))$$

taken over all sequences $(a_0, \dots, a_{n-1}) \in \{0, 1\}^n$.

Lemma 1. (see [PT], [Du2]) *Let (K, ψ) be a dynamically defined Cantor set with a Markov partition $\mathcal{P}$ and distortion $\mathrm{Dist}_\psi(K) = c$. Then*

$$e^{-c}\tau_L(\mathcal{P}) \leq \tau_L(K) \leq e^{c}\tau_L(\mathcal{P}), \qquad e^{-c}\tau_R(\mathcal{P}) \leq \tau_R(K) \leq e^{c}\tau_R(\mathcal{P}).$$

Lemma 2. (Left-right gap lemma, see [Mo], [Du2]) *Let (K^s, ψ^s), (K^u, ψ^u) be dynamically defined Cantor sets such that the intervals supporting K^s and K^u do intersect, K^s (resp. K^u) is not contained inside a gap of K^u (resp. K^s). If $\tau_L(K^s)\tau_R(K^u) > 1$ and $\tau_R(K^s)\tau_L(K^u) > 1$, then both Cantor sets intersect, $K^s \cap K^u \neq \emptyset$.*

Definition 5. *Define $\mathcal{F}$ to be the set of all maps $f : \mathbf{S}_0 \cup \mathbf{S}_1 \to \mathbb{R}^2$ such that:*

(1) $\mathbf{S}_0$ and $\mathbf{S}_1$ are compact sets, diffeomorphic to rectangles, with non-empty interior;

(2) f is a map of class C^2, in a neighborhood of $\mathbf{S}_0 \cup \mathbf{S}_1$, mapping this compact set diffeomorphically onto its image $f(\mathbf{S}_0) \cup f(\mathbf{S}_1)$;

(3) the maximal invariant set $\Lambda(f) = \cap_{n\in\mathbb{Z}} f^{-n}(\mathbf{S}_0 \cup \mathbf{S}_1)$ is a hyperbolic basic set conjugated to the Bernoulli shift $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$;

(4) $\mathcal{P} = \{\mathbf{S}_0, \mathbf{S}_1\}$ is a Markov partition for $f : \Lambda(f) \to \Lambda(f)$, in particular, f has two fixed points, $\mathbf{P}_0 \in \mathbf{S}_0$ and $\mathbf{P}_1 \in \mathbf{S}_1$, whose stable and unstable manifolds contain the boundaries of $\mathbf{S}_0$ and $\mathbf{S}_1$;

(5) both fixed points $\mathbf{P}_0$ and $\mathbf{P}_1$ have positive eigenvalues.

The action of f and f^{-1} respectively on the stable, and unstable, foliation of Λ,

$$\mathcal{F}^s = \{\text{connected comp. of } W^s(\Lambda) \cap (\mathbf{S}_0 \cup \mathbf{S}_1)\},$$

$$\mathcal{F}^u = \{\text{connected comp. of } W^u(\Lambda) \cap (f(\mathbf{S}_0) \cup f(\mathbf{S}_1))\},$$

can be described in the following way. Define

$$I^s_* = W^s_{loc}(\mathbf{P}_0) \cap \mathbf{S}_0 \qquad \text{and} \qquad I^u_* = W^u_{loc}(\mathbf{P}_0) \cap \mathbf{S}_0.$$

I^s_* and I^u_* are stable and unstable leaves of Λ respectively transversal to the foliation $\mathcal{F}^u$ and $\mathcal{F}^s$. Then the Cantor sets

$$K^s = \Lambda \cap I^u_* \qquad \text{and} \qquad K^u = \Lambda \cap I^s_*,$$

can be identified with the set of stable leaves of $\mathcal{F}^s$, respectively unstable leaves of $\mathcal{F}^u$. Define the projections $\pi_s : \Lambda \to K^s$ and $\pi_u : \Lambda \to K^u$ in the obvious way: $\pi_s(P)$ is the unique point in $W^s_{loc}(P) \cap I^u_*$, and similarly $\pi_u(P)$ is the unique point in $W^u_{loc}(P) \cap I^s_*$. The maps $\psi^s : K^s \to K^s$ and $\psi^u : K^u \to K^u$,

$$\psi^s = \pi_s \circ f \qquad \text{and} \qquad \psi^u = \pi_s \circ f^{-1},$$

describe the action of f, respectively f^{-1}, on stable, respectively unstable leaves of Λ. The pairs (K^s, ψ^s) and (K^u, ψ^u) are dynamically defined Cantor sets, topologically conjugated to the Bernoulli shift $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, with Markov partitions $\mathcal{P}^u = \{I^u_* \cap \mathbf{S}_0, I^u_* \cap \mathbf{S}_1\}$ and $\mathcal{P}^s = \{I^s_* \cap f(\mathbf{S}_0), I^s_* \cap f(\mathbf{S}_1)\}$.

Given a map $f \in \mathcal{F}$ we define the *left-right thickness* of $\Lambda(f)$ to be

$$\tau_{LR}(\Lambda) = \min\{\tau_L(K^s)\tau_R(K^u), \tau_L(K^u)\tau_R(K^s)\}.$$

In order to estimate this thickness we define the left-right thickness of $\mathcal{P}$ as

$$\tau_{LR}(\mathcal{P}) = \min\{\tau_L(\mathcal{P}^s)\tau_R(\mathcal{P}^u), \tau_L(\mathcal{P}^u)\tau_R(\mathcal{P}^s)\}.$$

If the distortion on both dynamically defined Cantor sets (K^s, ψ^s), (K^u, ψ^u) is small, say less than or equal to c, then it follows from Lemma 1 that

$$e^{-2c}\tau_{LR}(\mathcal{P}) \le \tau_{LR}(\Lambda) \le e^{2c}\tau_{LR}(\mathcal{P}).$$

0.4. **Duarte Distortion Theorem.**

Definition 6. *Given positive constants C^* along with small small ε and γ, define $\mathcal{F}(C^*, \varepsilon, \gamma)$ to be the class of all maps $f : S_0 \cup S_1 \to \mathbb{R}^2$, $f \in \mathcal{F}$, such that:*

(1) $diam\,(S_0 \cup S_1) \le 1, diam\,(f(S_0) \cup f(S_1)) \le 1;$

(2) the derivative of f, $Df_{(x,y)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are C^1-functions, satisfies all over $S_0 \cup S_1$

(a) $\det Df = ad - bc = 1,$

(b) $|d| < 1 < |a| \leq C^*/\varepsilon$,

(c) $|b|, |c| \leq \varepsilon(|a| - 1)$;

(3) the C^1*-functions on* $f(S_0) \cup f(S_1)$, $\tilde{a} = a \circ f^{-1}$, $\tilde{b} = b \circ f^{-1}$, $\tilde{c} = c \circ f^{-1}$ *and* $\tilde{d} = d \circ f^{-1}$, *i.e.* $Df^{-1}_{(x,y)} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}$, *satisfy*

(a) $\left|\frac{\partial \tilde{b}}{\partial x}\right| = \left|\frac{\partial \tilde{d}}{\partial y}\right|, \left|\frac{\partial \tilde{b}}{\partial y}\right|, \left|\frac{\partial \tilde{c}}{\partial x}\right|, \left|\frac{\partial \tilde{a}}{\partial x}\right| = \left|\frac{\partial \tilde{c}}{\partial y}\right| \leq \gamma(|\tilde{a}| - 1)$,

(b) $\left|\frac{\partial a}{\partial y}\right| = \left|\frac{\partial b}{\partial x}\right|, \left|\frac{\partial b}{\partial y}\right|, \left|\frac{\partial c}{\partial x}\right|, \left|\frac{\partial c}{\partial y}\right| = \left|\frac{\partial d}{\partial x}\right| \leq \gamma(|\tilde{a}| - 1)$,

(c) $\left|\frac{\partial \tilde{a}}{\partial y}\right|, \left|\frac{\partial \tilde{d}}{\partial x}\right| \leq \gamma|\tilde{a}|(|\tilde{a}| - 1)$,

(d) $\left|\frac{\partial a}{\partial x}\right|, \left|\frac{\partial d}{\partial y}\right| \leq \gamma|\tilde{a}|(|\tilde{a}| - 1)$;

(4) the variation of $\log|a(x, y)|$ *in each rectangle* S_i *is less or equal to* $\gamma(1 - \alpha_i^{-1})$, *where* $\alpha_i = \max_{(x,y)\in S_i} |a(x, y)|$;

(5) finally, the gap sizes satisfy:

$$\mathit{dist}(S_0, S_1) \geq \frac{\varepsilon}{\gamma} \qquad \text{and} \qquad \mathit{dist}(f(S_0), f(S_1)) \geq \frac{\varepsilon}{\gamma}.$$

Remark 3. *For* $C^* = 2$ *this definition coincides with Definition 4 in* [Du2].

The nice feature of the maps from $\mathcal{F}(C^*, \varepsilon, \gamma)$ is that the stable and unstable foliations have small uniformly bounded distortion, as the following theorem shows.

Theorem 3. *For a given* $C^* > 0$ *and all small enough* $\varepsilon > 0$ *and* $\gamma > 0$, *given* $f \in \mathcal{F}(C^*, \varepsilon, \gamma)$, *the basic set* $\Lambda(f)$ *gives dynamically defined Cantor sets* (K^u, ψ^u) *and* (K^s, ψ^s) *with small distortion, bounded by* $D(C^*, \varepsilon, \gamma) = 4(C^*+3)\gamma+2\varepsilon$. *In particular,*

$$\begin{aligned} e^{-D(C^*,\varepsilon,\gamma)}\tau_L(\mathcal{P}^s) \leq \tau_L(K^s(f)) \leq e^{D(C^*,\varepsilon,\gamma)}\tau_L(\mathcal{P}^s), \\ e^{-D(C^*,\varepsilon,\gamma)}\tau_R(\mathcal{P}^s) \leq \tau_R(K^s(f)) \leq e^{D(C^*,\varepsilon,\gamma)}\tau_R(\mathcal{P}^s), \\ e^{-D(C^*,\varepsilon,\gamma)}\tau_L(\mathcal{P}^u) \leq \tau_L(K^u(f)) \leq e^{D(C^*,\varepsilon,\gamma)}\tau_L(\mathcal{P}^u), \\ e^{-D(C^*,\varepsilon,\gamma)}\tau_R(\mathcal{P}^u) \leq \tau_R(K^u(f)) \leq e^{D(C^*,\varepsilon,\gamma)}\tau_R(\mathcal{P}^u). \end{aligned}$$

Remark 4. *Again, for* $C^* = 2$ *this Theorem coincides with Theorem 2 in* [Du2]. *Notice that conditions (2b) and (2c) of definition 6 imply that* $|b|, |c| \leq C^*$, *and for* $C^* = 2$ *this gives an unreasonable restriction on the class of maps that could be considered. We will need to apply Theorem 3 for a map which belongs to the class* $\mathcal{F}(C^*, \varepsilon, \gamma)$ *with larger value of* C^*, *see Proposition 20.*

Proof of Theorem 3. The straightforward repetition of the proof of Theorem 2 in [Du2] with the necessary adjustments needed to take the constant C^* into account proves Theorem 3. The only place in the proof of Theorem 2 from [Du2] where the condition $|a| \leq \frac{2}{\varepsilon}$ is used is the inequality (3) from Lemma 4.2. If we use the inequality $|a| \leq \frac{C^*}{\varepsilon}$ instead, 6γ should be replaced by $\frac{3}{2}(C^* + 2)$ there. Due to this change, in Lemma 4.1 one should take $2(C^* + 2)\gamma$ instead of 8γ as an upper bound of Lipschitz seminorm $Lip(\sigma^s)$ and $Lip(\sigma^u)$ of functions σ^s and σ^u that describe stable and unstable foliations. This leads to similar changes in the statement of Lemma 4.4 and in the estimate of the distortion. Finally we get the above statement. Theorem 3 is proved.

0.5. **The definition of the Hausdorff dimension.**

Let us recall the definition of the Hausdorff dimension. Let $K \subset \mathbb{R}$ be a Cantor set and $\mathcal{U} = \{U_i\}_{i\in I}$ a finite covering of K by open intervals in $\mathbb{R}$. We define the diameter $\mathrm{diam}(\mathcal{U})$ of $\mathcal{U}$ as the maximum of $|U_i|$, $i \in I$, where $|U_i|$ denotes the length of U_i. Define $H_\alpha(\mathcal{U}) = \sum_{i\in I} |U_i|^\alpha$. Then the *Hausdorff α-measure* of K is

$$m_\alpha(K) = \lim_{\varepsilon\to 0}\left(\inf_{\mathcal{U} \text{ covers } K,\ \mathrm{diam}(\mathcal{U})<\varepsilon} H_\alpha(\mathcal{U})\right).$$

It is not hard to see that there is a unique number, the *Hausdorff dimension* of K, denoted by $\dim_H(K)$, such that for $\alpha < \dim_H(K)$, $m_\alpha(K) = \infty$ and for $\alpha > \dim_H(K)$, $m_\alpha(K) = 0$.

0.6. **Large thickness implies large Hausdorff dimension.**

Proposition 3. *Consider a Cantor set K, denote $\tau_L = \tau_L(K)$ and $\tau_R = \tau_R(K)$, and let d be the solution of the equation*

$$\tau_L^d + \tau_R^d = (1 + \tau_L + \tau_R)^d.$$

Then $\dim_H(K) \geq d$.

Remark 5. *One can consider Proposition 3 as a generalization of the Proposition 5 from Chapter 4.2 in* [PT], *where the relation between the usual thickness and the Hausdorff dimension of a Cantor set was established.*

Proof of the Proposition 3. We will need the following elementary Lemma.

Lemma 4. *For any $d \in (0,1)$ and $\tau_L, \tau_R > 0$*

$$\min\left\{x^d + y^d \mid x \geq 0, y \geq 0, x + y \leq 1, x \geq \tau_L(1 - x - y), y \geq \tau_R(1 - x - y)\right\} = \\ = \left(\frac{\tau_L}{1 + \tau_L + \tau_R}\right)^d + \left(\frac{\tau_R}{1 + \tau_L + \tau_R}\right)^d.$$

The proof of Lemma 4 is left to the reader.

We show that $H_d(\mathcal{U}) \geq (\text{diam } K)^d$ for every finite open covering $\mathcal{U}$ of K, which clearly implies the proposition. We can assume that $\mathcal{U}$ is a covering with disjoint intervals. This is no restriction because whenever two elements of $\mathcal{U}$ have nonempty intersection we can replace them by their union, getting in this way a new covering $\mathcal{V}$ such that $H_d(\mathcal{V}) \leq H_d(\mathcal{U})$. Note that, since $\mathcal{U}$ is an open covering of K, it covers all but finite number of gaps of K. Let U, a gap of K, have minimal order among the gaps of K which are not covered by $\mathcal{U}$. Let C^L and C^R be that bridges of K at the boundary points of U.

By construction there are A^L, $A^R \in \mathcal{U}$ such that $C^L \subset A^L$ and $C^R \subset A^R$. Take the convex hall A of $A^L \cup A^R$. Then

$$|A^L| \geq |C^L| \geq \tau_L \cdot |U| \geq \tau_L(|A| - |A^L| - |A^R|)$$

and

$$|A^R| \geq |C^R| \geq \tau_R \cdot |U| \geq \tau_R(|A| - |A^L| - |A^R|).$$

Or, equivalently,

$$\frac{|A^L|}{|A|} \geq \tau_L \left(1 - \frac{|A^L|}{|A|} - \frac{|A^R|}{|A|}\right)$$

and

$$\frac{|A^R|}{|A|} \geq \tau_R \left(1 - \frac{|A^L|}{|A|} - \frac{|A^R|}{|A|}\right).$$

Lemma 4 now implies that

$$\left(\frac{|A^L|}{|A|}\right)^d + \left(\frac{|A^R|}{|A|}\right)^d \geq \left(\frac{\tau_L}{1+\tau_L+\tau_R}\right)^d + \left(\frac{\tau_R}{1+\tau_L+\tau_R}\right)^d = 1,$$

and $|A^L|^d + |A^R|^d \geq |A|^d$. This means that the covering $\mathcal{U}_1$ of K obtained by replacing A^L and A^R by A in $\mathcal{U}$ is such that $H_d(\mathcal{U}_1) \leq H_d(\mathcal{U})$. Repeating the argument we eventually construct $\mathcal{U}_k$, a covering of the convex hall of K with $H_d(\mathcal{U}_k) \leq H_d(\mathcal{U})$. Since we must have $H_d(\mathcal{U}_k) \geq (\text{diam } K)^d$, this ends the proof.

Proposition 3 can be used to find an explicit estimate of the Hausdorff dimension via one-sided thicknesses. In particular, when one of the one-sided thicknesses is very large and another one is small, the following Proposition gives an estimate that is good enough for our purposes.

Proposition 5. *Denote by τ_L and τ_R the left and right thicknesses of the Cantor set $K \subset \mathbb{R}$. Then*

$$dim_H K > \max\left(\frac{\log\left(1+\frac{\tau_R}{1+\tau_L}\right)}{\log\left(1+\frac{1+\tau_R}{\tau_L}\right)}, \quad \frac{\log\left(1+\frac{\tau_L}{1+\tau_R}\right)}{\log\left(1+\frac{1+\tau_L}{\tau_R}\right)}\right).$$

Proof of Proposition 5. We will use the following Lemma.

Lemma 6. *Assume that for some $x, y > 0$, $x + y < 1$, and some $d_1, d_2 \in (0,1)$ the following relations hold:*

$$y = (1-x)^{\frac{1}{d_1}}, \qquad x^{d_2} + y^{d_2} = 1.$$

Then $d_2 > d_1$.

Indeed, $(1-x)^{\frac{1}{d_1}} = y = (1-x^{d_2})^{\frac{1}{d_2}} < (1-x)^{\frac{1}{d_2}}$, so due to our choice of x we have $d_2 > d_1$.

Let us apply Lemma 6 to $x = \frac{\tau_L}{1+\tau_L+\tau_R}$ and $y = \frac{\tau_R}{1+\tau_L+\tau_R}$. If $y = (1-x)^{\frac{1}{d_1}}$, $x^{d_2}+y^{d_2} = 1$ for some $d_1, d_2 \in (0,1)$ then by Proposition 3 we have

$$\dim_H K \geq d_2 > d_1 = \frac{\log(1-x)}{\log y} = \frac{\log\left(1 - \frac{\tau_L}{1+\tau_L+\tau_R}\right)}{\log\left(\frac{\tau_R}{1+\tau_L+\tau_R}\right)} = \frac{\log\left(1 + \frac{\tau_L}{1+\tau_R}\right)}{\log\left(1 + \frac{1+\tau_L}{\tau_R}\right)}.$$

In a similar way one can show that $\dim_H K > \frac{\log\left(1+\frac{\tau_R}{1+\tau_L}\right)}{\log\left(1+\frac{1+\tau_R}{\tau_L}\right)}$. Proposition 5 is proved.

Remark 6. *Assume that $\tau_R \sim \frac{1}{\lambda-1}$, $\tau_L \sim (\lambda-1)^\nu$. Then*

$$\lim_{\lambda\to 1+0} \frac{\log\left(1+\frac{\tau_R}{1+\tau_L}\right)}{\log\left(1+\frac{1+\tau_R}{\tau_L}\right)} = \lim_{\lambda\to 1+0} \frac{\log\left(1+\frac{\frac{1}{\lambda-1}}{1+(\lambda-1)^\nu}\right)}{\log\left(1+\frac{1+\frac{1}{\lambda-1}}{(\lambda-1)^\nu}\right)} = \frac{1}{1+\nu}.$$

So if ν is small enough and λ is close to one, then $\dim_H K$ is close to 1.

Moreover, if $\tau_R \sim \frac{1}{\lambda-1}$, $\tau_L \sim \log(\lambda-1)^{-1}$ then

$$\lim_{\lambda\to 1+0} \frac{\log\left(1+\frac{\tau_R}{1+\tau_L}\right)}{\log\left(1+\frac{1+\tau_R}{\tau_L}\right)} = 1,$$

and therefore $\dim_H K \to 1$ as $\lambda \to 1$.

0.7. **Uniqueness of the area preserving quadratic family.**

In [H], [F] it was established that up to the change of parameter and coordinates there exists only one one-parameter area preserving quadratic family with some conditions on the fixed points (Henon family). In particular, we can consider the family

$$F_\varepsilon : (x,y) \mapsto (x+y-x^2+\varepsilon,\ y-x^2+\varepsilon) \qquad (2)$$

instead of (1). In this form it is a partial case of a so called *generalized standard family*, and it was considered in [G1].

0.8. **Rescaling and the family of maps close to identity.**

Let us consider the following family of the affine coordinate changes:

$$\Upsilon_\delta \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} \delta^2 \\ 0 \end{pmatrix} + \begin{pmatrix} \delta^2 & 0 \\ 0 & \delta^3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $\delta = \varepsilon^{\frac{1}{4}}$. Then

$$\Upsilon_\delta^{-1} \circ F_{\delta^4} \circ \Upsilon_\delta \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + \delta v \\ v + \delta(2u - u^2) \end{pmatrix} + \delta^2 \begin{pmatrix} 2u - u^2 \\ 0 \end{pmatrix},$$

Now we have a family of maps close to identity. For each of these maps the origin is a saddle with eigenvalues

$$\lambda_1 = 1 + \delta^2 + \sqrt{\delta^4 + 2\delta^2} = 1 + \sqrt{2}\delta + O(\delta^2) > 1,$$

$$\lambda_2 = \lambda_1^{-1} = 1 + \delta^2 - \sqrt{\delta^4 + 2\delta^2} = 1 - \sqrt{2}\delta + O(\delta^2) < 1.$$

Set $h = \log \lambda_1$. By definition $h = \sqrt{2}\delta + O(\delta^2)$, and δ can be given by implicit function of h. Define the following (rescaled and reparametrized) family

$$\mathfrak{F}_h : (u,v) \mapsto (u,v) + \delta(v, 2u - u^2) + \delta^2(2u - u^2, 0). \tag{3}$$

0.9. **Birkhoff normal form.**

First of all let us recall that the real analytic area preserving diffeomorphism of a two dimensional domain in a neighborhood of a saddle with eigenvalues (λ, λ^{-1}) by an analytic change of coordinate can be reduced to the Birkhoff normal form ([S], see also [SM]):

$$N(x,y) = (\Delta(xy)x, \Delta^{-1}(xy)y), \tag{4}$$

where $\Delta(xy) = \lambda + a_1 xy + a_2(xy)^2 + \ldots$ is analytic.

We need a generalization of this Birkhoff normal form for one-parameter families.

Theorem 4. ([FS], *Proposition 3.1;* [Du2], *Proposition 6.1) There exists a neighborhood U of the origin such that for all $h \in (0, h_0)$ there exists a coordinate change C_h in U with the following properties:*

1. If $N_h = C_h \mathfrak{F}_h C_h^{-1}$ then $N_h(u,v) = (\Delta_h(uv)u, \Delta_h^{-1}(uv)v)$, where $\Delta_h(uv) = \lambda(h) + a_1(h)uv + a_2(h)(uv)^2 + \ldots$ is analytic.

2. C^3-norms of the coordinate changes C_h are uniformly bounded with respect to the parameter h.

3. $\Delta_h(s) \geq 1$ is a smooth function of s and h.

Remark 7. *The second property is not formulated explicitly in* [FS] *but it easily follows from Cauchy estimates. Indeed, it follows from the proof there that the map C_h is analytic and radius of convergence of the corresponding series is uniformly bounded from below.*

Also we will need the following property of the parametric Birkhoff normal form for the family $\mathfrak{F}_h$.

Lemma 7. *For some constant $C > 0$ and small enough $h_0 > 0$ and $s_0 > 0$ the following holds. For all $h \in [0, h_0)$ and $s \in [0, s_0)$*

1. $\log \Delta_h(s) \geq C^{-1}h$,
2. $|\Delta_h'(s)| \leq Ch$,
3. $|\Delta_h''(s)| \leq Ch$.

Remark 8. *This Lemma is similar to Lemma 6.3 from* [Du2]*, but in our case we have one, not two parameter family, and therefore those two statements are essentially different.*

Proof of Lemma 7. Consider $g(s, h) = \log \Delta_h(s)$. We have $g(s, 0) = 0$, $g(0, h) = h$, and g is a smooth function of (s, h). This implies that for small enough $s_0 > 0$, $h_0 > 0$ and large $C > 0$ we have $g(s, h) \geq C^{-1}h$ for all $s \in [0, s_0]$ and $h \in [0, h_0]$.

From the explicit form of the family $\mathfrak{F}_h$ (3) we see that $\mathfrak{F}_h \to \mathrm{Id}$ as $h \to 0$ in C^r-norm for every $r \in \mathbb{N}$. Since C^3-norms of C_h and C_h^{-1} are uniformly bounded, this implies that $N_h \to \mathrm{Id}$ in C^2-norm as $h \to 0$. In particular,

$$DN_h(x, y) = \begin{pmatrix} \Delta_h(xy) + \Delta_h'(xy)xy & x^2\Delta_h'(xy) \\ -\frac{\Delta_h'(xy)y^2}{\Delta_h^2(xy)} & \Delta_h^{-1}(xy) - \frac{\Delta_h'(xy)xy}{\Delta^2(xy)} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as $h \to 0$ uniformly in $(x, y) \in U$. This implies that $\Delta_h'(s) \to 0$ as $h \to 0$ uniformly in $s \in [s_1, s_0]$ for every $s_1 \in (0, s_0)$. Also $\Delta_0(s) = 1$ for every $h \in [0, h_0)$, so $\Delta_0'(s) = 0$. Since $\Delta_h'(s)$ is a continuous function, this implies that $\Delta_h'(s) \to 0$ as $h \to 0$ uniformly in $s \in [0, s_0]$. Since $\Delta_h'(s)$ is a smooth function of (s, h), this implies that $|\Delta_h'(s)| \leq Ch$ if $C > 0$ is large enough. Similarly one can show that $|\Delta_h''(s)| \leq Ch$. Lemma 7 is proved.

0.10. **Gelfreich normal form.**

The family $\mathfrak{F}_h$ is closely related to the conservative vector field

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2x - x^2. \end{cases} \tag{5}$$

Namely, due to Theorems A and A' from [FS1] (see also Proposition 5.1 from [FS]) the separatrix phase curve of the vector field (5) (let us denote it by σ) gives a good approximation of some finite pieces of $W^s(0,0)$ and $W^u(0,0)$. Denote by $\widetilde{\sigma}$ a segment of separatrix σ that contains some points $P^u \in W^u_{loc}(0,0) \cap U$ and $P^s \in W^s_{loc}(0,0) \cap U$ (see Fig. ?) and by V a neighborhood of $\widetilde{\sigma}$. Denote by $\widetilde{W^s_h}(0,0)$ the finite piece of $W^s_h(0,0)$ between the points where $W^s_h(0,0)$ leaves U for the first time and the first point where $W^s_h(0,0)$ returns to U again. Define $\widetilde{W^u_h}(0,0)$ in a similar way. Then $\widetilde{W^s_h}(0,0)$ and $\widetilde{W^u_h}(0,0)$ are always (for all $h \in (0,h_0)$) in V.

The restriction of the map $\mathfrak{F}_h$ on the local separatrix $W^u_{\mathfrak{F}_h}(0)$ is conjugated with a multiplication $\xi \mapsto \lambda\xi$, $\xi \in (\mathbb{R},0)$. Let us call a parameter t on $W^u_{\mathfrak{F}_h}(0)$ *standard* if it is obtained by a substitution of e^t instead of ξ into the conjugating function. Such a parametrization is defined up to a substitution $t \mapsto t + const$.

Denote $\Pi_{r_0,E_0} = \{(t,E) \in \mathbb{R}^2 \mid |t| < r_0, |E| < E_0\}$.

The following Lemma follows from Theorem 4 in [G3].

Lemma 8. *There are neighborhood V of the segment of σ between points P^u and P^s and constants r_0 and E_0 such that for some $h_0 > 0$ and all $h \in (0,h_0)$ there exists a map $\Psi_h : \Pi_{r_0,E_0} \to \mathbb{R}^2$ with the following properties:*

1. *$\Psi_h(\Pi_{r_0,E_0}) \supset V$;*
2. *Ψ_h is real analytic;*
3. *Ψ_h is area preserving;*
4. *Ψ_h conjugates the map $\mathfrak{F}_h$ with the shift $(t,E) \mapsto (t+h,E)$;*
5. *$\Psi_h^{-1}(\widetilde{W^u_h}) = \{E = 0\}$, and t gives a standard parametrization of the unstable manifold;*
6. *C^3-norms of Ψ_h and Ψ_h^{-1} are uniformly bounded with respect to $h \in (0,h_0)$.*

Remark 9. *In* [G3] *uniform estimates of first derivatives only are included to the statement, but the uniform boundedness of the second and third derivatives easily follows from the Cauchy type estimates since the radius of convergence of the power series that represent Ψ_h is uniformly bounded from below. We are grateful to V.Gelfreich for this remark.*

0.11. **Splitting of separatrices.**

In [G1] for the initial family F_ε (2) the form of W^s_h is described in some normalized coordinates (see also Theorem 7.1 and Proposition 7.2 from [G2]).

Denote $\mathcal{G}_{\mathbb{R}} = \{(t, E) \in \mathbb{R}^2 \mid |t| \le 10h, |E| \le h^9\}$.

The formal statement is the following:

Theorem 5. *([G1], [G2]) There exists h_0 such that for all $h \in (0, h_0)$ there exists a map $\Phi_h : \mathcal{G}_{\mathbb{R}} \to \mathbb{R}^2$ such that*

1. *Φ_h is real analytic;*
2. *Φ_h is area preserving;*
3. *Φ_h conjugates the map $F_{\varepsilon(h)}$ with the shift $(t, E) \mapsto (t + h, E)$;*
4. *$\Phi_h^{-1}(\widetilde{W}_h^u) = \{E = 0\}$, and t gives a standard parametrization of the unstable manifold;*
5. *the second projection of the inverse map $E = pr_2 \circ \Phi^{-1}$ has the derivatives of the first order bounded by $const \cdot h^{-1}$ and the derivatives of the second order bounded by $const \cdot h^{-10}$; the first derivatives of the first projection are bounded by $const \cdot h^{-2}$;*
6. *$\Phi_h^{-1}(\widetilde{W}_h^s)$ is a graph of some real-analytic h-periodic function $\Theta(t)$;*
7. *The function $\Theta(t)$ has the following form:*

$$\Theta(t) = 2|\Theta_1|\, h^{-1} e^{-2\pi^2/h} \sin \frac{2\pi t}{h} + O(h e^{-2\pi^2/h}); \tag{6}$$

$$\dot{\Theta}(t) = 4\pi|\Theta_1|\, h^{-2} e^{-2\pi^2/h} \cos \frac{2\pi t}{h} + O(e^{-2\pi^2/h}). \tag{7}$$

Here $|\Theta_1|$ is a constant that does not depend on h.

Remark 10. $|\Theta_1| \neq 0$*, see* [GS] *for the proof.*

Remark 11. *Also the following asymptotic formula holds for the second derivative of the function $\Theta(t)$ ([G]):*

$$\ddot{\Theta}(t) = -8\pi^2|\Theta_1|\, h^{-3} e^{-2\pi^2/h} \sin \frac{2\pi t}{h} + O(h^{-1} e^{-2\pi^2/h}). \tag{8}$$

Remark 12. *In* [G2] *complex values of t and E were considered (which is an essential part of the proof). Since we do not use complexified invariant manifolds, we state only the "real" part of the claim here.*

We would like to obtain formulas similar to (6) in the previous Theorem for the stable manifold for the family $\mathfrak{F}_h$. The following statement holds:

Proposition 9. *Stable manifold $\Psi_h^{-1}(\widetilde{W}_h^s)$ can be represented as a graph of a real-analytic h-periodic function $\Theta^*(t)$ such that*

$$\Theta^*(t) = 8\sqrt{2}|\,\Theta_1|\,h^{-6}e^{-2\pi^2/h}\sin\frac{2\pi t}{h} + O(h^{-5}e^{-2\pi^2/h}); \tag{9}$$

$$\dot{\Theta}^*(t) = 16\sqrt{2}\pi|\,\Theta_1|\,h^{-7}e^{-2\pi^2/h}\cos\frac{2\pi t}{h} + O(h^{-6}e^{-2\pi^2/h}); \tag{10}$$

$$\ddot{\Theta}^*(t) = -32\sqrt{2}\pi^2|\Theta_1|\,h^{-8}e^{-2\pi^2/h}\sin\frac{2\pi t}{h} + O(h^{-7}e^{-2\pi^2/h}). \tag{11}$$

Remark 13. *To simplify the notation define the function*

$$\mu(h) = 16\sqrt{2}\pi|\,\Theta_1|\,h^{-7}\exp{(-2\pi^2/h)}. \tag{12}$$

Notice that the angle between $\widetilde{W}_h^u$ and $\widetilde{W}_h^s$ at the homoclinic point in the normalized coordinates is equal to $\mu(h)(1+O(h))$. Formulas (9) can be now rewritten in the following way:

$$\Theta^*(t) = \frac{1}{2\pi}h\mu(h)\sin\frac{2\pi t}{h} + O(h^2\mu(h)),\quad \dot{\Theta}^*(t) = \mu(h)\cos\frac{2\pi t}{h} + O(h\mu(h)),$$

$$\ddot{\Theta}^*(t) = -2\pi h^{-1}\mu(h)\sin\frac{2\pi t}{h} + O(\mu(h)).$$

The idea of proof of Proposition 9. The only difference with respect to the formula (6) is due to the fact that transformation to the close-to-identity form is not area-preserving, so the E coordinates gains a factor equal to the Jacobian of the transformation if we want a symplectic system of coordinates. So the first step is to scale E the original E in such a way that new (T,E) are symplectic coordinates for the close-to-identity map.

Then the proof should follow the following lines. Assume you have (9) in some coordinates (T,E) and want to show that the same formula holds in any other coordinates, say (t,e) such that

1) (t,0) corresponds to the unstable manifold, which implies $(T,0)\mapsto(t,e)=(T,0)$

2) $(T,E)\mapsto(t,e)$ is area-preserving

3) the change of coordinates commutes with the translation $(T,E)\mapsto(t+h,E)$

4) the derivatives of $(x,y)\mapsto(t,e)$ do not grow too fast as $h\to 0$ (e.g. uniformly bounded)

1)-3) imply

$(t,e) = (T+O(E), E+O(E^2))$,

where both O are h-periodic in T. Of course, the constants in the O estimates depend from h. Now take into account 4) – the coordinate change $(T, E) \to (t, e)$ comes as a composition of two changes ($(T, e) \to$ the original (x, y) and back $\to (t, e)$). Therefore the constants do not grow faster than say h^{-10}.

Now take the image of the graph given by (9). In its neighborhood the map $(T, E) \to (t, e)$ is very close to identity as $E = \theta$ – exponentially small. Consequently, the form of the graph is the same in the new coordinates (t, e).

0.12. **Construction of the domain for the first return map.**

Let q_h^u be the closest to $P^u \in \sigma$ point of intersection of $\widetilde{W}_h^u$ and $\widetilde{W}_h^s$. Consider a finite sequence of images of q_h^u under the map $\mathfrak{F}_h$ that belong to the neighborhood V, $\{q_h^u, \mathfrak{F}(q_h^u), \mathfrak{F}^2(q_h^u), \dots\}$. Let q_h^s be the point of this sequence closest to the point $P^s \in \sigma$. Define $k(h) \in \mathbb{N}$ by $\mathfrak{F}_h^{k(h)}(q_h^u) = q_h^s$. Take the vector $v = (1, 0) \in T_{q_h^u} U$ and consider the vector $w = (w_1, w_2) = D(C_h \circ \Psi_h \circ H^{k(h)} \circ C_h^{-1} \circ \Psi_h^{-1})v \in T_{q_h^s} U$. Without loss of generality we can assume that $w_1 > 0$ (otherwise just take the homoclinic point between q_h^u and $\mathfrak{F}(q_h^u)$ instead of q_h^u). Scaling, if necessary, we can assume that in the Birkhoff normalizing coordinates we have $q_h^u = (1, 0)$, $q_h^s = (0, 1)$.

Fix small $\nu > 0$. Recall that $\lambda = \Delta_h(0) = e^h$. Set

$$n = \left[-\frac{\log(\mu(h) h^{1+\nu})}{2h} \right]. \tag{13}$$

Due to this choice $\lambda^{-2n} \approx \mu(h) h^{1+\nu}$. More precisely, $\lambda^{-2n} \in [\mu(h) h^{1+\nu}, \lambda^2 \mu(h) h^{1+\nu})$.

Remark 14. *Notice that this choice of n for $\nu = \frac{1}{2}$ is analogous to the formula (7) in* [Du2].

Define the following lines:

$$\tau_{(1,0)}^+ = \{x = \lambda^{\frac{1}{10}}\}, \qquad \tau_{(1,0)}^- = \{x = \lambda^{-\frac{1}{10}}\},$$
$$\tau_{(0,1)}^+ = \{y = \lambda^{\frac{1}{10}}\}, \qquad \tau_{(0,1)}^- = \{y = \lambda^{-\frac{1}{10}}\}.$$

Denote by S the square formed by $W_{loc}^s(0), W_{loc}^u(0), N_h^n(\tau_{(0,1)}^+)$ and $N_h^{-n}(\tau_{(1,0)}^+)$. The bottom and left edges of S has the size

$$l = \lambda^{-n+\frac{1}{10}}. \tag{14}$$

Notice that since DN_h is close to the linear map $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, the curve $N_h^n(\tau_{(0,1)}^+)$ (resp., $N_h^{-n}(\tau_{(1,0)}^+)$) is C^1-close to a horizontal (resp., vertical) line. Denote by R^u and R^s the rectangles formed by $W^u(0)$, $\tau_{(1,0)}^+$, $\tau_{(1,0)}^-$ and $N_h^{2n}(\tau_{(0,1)}^+)$, and by $W^s(0)$,

$\tau^+_{(0,1)}$, $\tau^-_{(0,1)}$ and $N_h^{-2n}(\tau^+_{(1,0)})$, respectively. Notice that $R^u = N_h^n(S) \cap \{x \geq \lambda^{-\frac{1}{10}}\}$ and $R^s = N_h^{-n}(S) \cap \{y \geq \lambda^{-\frac{1}{10}}\}$.

Denote by R^* the intersection (see Fig. ?):

$$R^* = H^{k(h)} \circ C_h^{-1} \circ \Psi_h^{-1}(R^u) \cap C_h^{-1} \circ \Psi_h^{-1}(R^s).$$

Now consider the rectangles

$$S_0 = S \cap N_h^{-1}(S) \quad \text{and} \quad S_1 = N_h^{-n} \circ \Psi_h \circ C_h \circ H^{-k(h)}(R^*) \tag{15}$$

and define the first return map

$$T(x,y) = \begin{cases} N_h(x,y), & \text{if } (x,y) \in S_0; \\ N_h^n \circ C_h \circ \Psi_h \circ H^{k(h)} \circ C_h^{-1} \circ \Psi_h^{-1} \circ N_h^n(x,y), & \text{if } (x,y) \in S_1. \end{cases}$$

0.13. **Renormalization.**

We are going to prove that the map T has a hyperbolic invariant set in S and to investigate its properties (Hausdorff dimension, lateral thicknesses) with respect to the parameter h. In order to get a family of maps in the square of a fixed size we rescale the map T. Namely, we consider a map $\rho : S \to [0,2] \times [0,2]$, $\rho(x,y) = (l^{-1}x, l^{-1}y)$.

Denote $\rho(S_0) = \tilde{S}_0, \rho(S_1) = \tilde{S}_1$, and define

$$\tilde{T} : \tilde{S}_0 \cup \tilde{S}_1 \to [0,2] \times [0,2] \quad \text{by} \quad \tilde{T} = \rho \circ T \circ \rho^{-1}. \tag{16}$$

One can obtain the following two Lemmas by a straightforward calculation using Lemma 7 and the mean value theorem.

Lemma 10. *For $(x,y) \in \widetilde{S}_0$ we have*

$$D\widetilde{T}|_{\widetilde{S}_0}(x,y) = \begin{pmatrix} \Delta_h(l^2xy) + \Delta_h'(l^2xy)l^2xy & \Delta_h'(l^2xy)l^2x^2 \\ -\frac{\Delta_h'(l^2xy)l^2y^2}{\Delta_h(l^2xy)} & \Delta_h^{-1}(l^2xy) - \frac{\Delta_h'(l^2xy)l^2xy}{\Delta_h(l^2xy)} \end{pmatrix} = \tag{17}$$
$$= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} + \begin{pmatrix} O(\mu(h)h^{2+\nu}) & O(\mu(h)h^{2+\nu}) \\ O(\mu(h)h^{2+\nu}) & O(\mu(h)h^{2+\nu}) \end{pmatrix},$$

where $\lambda = e^h$.

Notice that if $(x,y) \in R^u$, $(x,y) \in S$, or $(x,y) \in R^s$ then $|xy| \leq 4\lambda^{-2n}$. If $(x,y) \in S$, then also $x^2 \leq 4\lambda^{-2n}$ and $y^2 \leq 4\lambda^{-2n}$.

Lemma 11. *For $(x,y)\in R^u$ we have*

(18) $$DN_h^n(x,y)=\begin{pmatrix}\Delta_h^n(xy)+n\Delta_h^{n-1}(xy)\Delta_h'(xy)xy & n\Delta_h^{n-1}(xy)\Delta_h'(xy)x^2\\ -\frac{n\Delta_h'(xy)y^2}{\Delta_h^{n+1}(xy)} & \Delta_h^{-n}(xy)-\frac{n\Delta_h'(xy)xy}{\Delta_h^{n+1}(xy)}\end{pmatrix}=$$
$$=\begin{pmatrix}\lambda^n & 0\\ 0 & \lambda^{-n}\end{pmatrix}+\begin{pmatrix}O(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|) & O((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|)\\ O(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|) & O((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|)\end{pmatrix}.$$

For $(x,y)\in S$ we have

(19) $$DN_h^n(x,y)=$$
$$=\begin{pmatrix}\lambda^n & 0\\ 0 & \lambda^{-n}\end{pmatrix}+\begin{pmatrix}O(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|) & O(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|)\\ O((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|) & O((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|)\end{pmatrix}.$$

0.14. **Cone condition.**

The coordinate changes $C_h\circ\Psi_h$ and $\Psi_h^{-1}\circ C_h^{-1}$ have uniformly bounded C^3-norms. Assume that their C^3-norms are bounded by some constant C_0.

Let us introduce the following cone fields in $\tilde S_0\cup\tilde S_1$:

(20) $$K^u(x,y)=\{\bar v=(v_1,v_2)\in T_{(x,y)}\tilde S_i \mid |v_1|>0.001C_0^{-6}h^{-1-\nu}|v_2|\},\quad\text{and}$$

(21) $$K^s(x,y)=\{\bar v=(v_1,v_2)\in T_{(x,y)}\tilde S_i \mid |v_2|>0.001C_0^{-6}h^{-1-\nu}|v_1|\}.$$

Lemma 12. (Cone condition for $\tilde S_0$) *Assume that h is small enough.*

For every vector $\bar v\in K^u(x,y)$, $(x,y)\in\tilde S_0$, we have $D\tilde T_{(x,y)}(\bar v)\in K^u(\tilde T(x,y))$, and if $D\tilde T_{(x,y)}(\bar v)=\bar w\equiv(w_1,w_2)$ then $|w_1|\ge\lambda^{0.9}|v_1|$.

For every vector $\bar v\in K^s(x,y)$, $(x,y)\in\tilde T(\tilde S_0)$, we have $D\tilde T^{-1}_{(x,y)}(\bar v)\in K^s(\tilde T^{-1}(x,y))$, and if $D\tilde T^{-1}_{(x,y)}(\bar v)=\bar w\equiv(w_1,w_2)$ then $|w_2|\ge\lambda^{0.9}|v_2|$.

Proof of Lemma 12. This follows directly from Lemma 10.

Lemma 13. (Cone condition for $\tilde S_1$) *Assume that h is small enough.*

For every vector $\bar v\in K^u(x,y)$, $(x,y)\in\tilde S_1$, we have $D\tilde T_{(x,y)}(\bar v)\in K^u(\tilde T(x,y))$, and if $D\tilde T_{(x,y)}(\bar v)=\bar w\equiv(w_1,w_2)$ then $|w_1|\ge 0.01C_0^{-4}h^{-1-\nu}|v_1|$ and $|\bar w|\le 25C_0^4h^{-1-\nu}|\bar v|$.

For every vector $\bar v\in K^s(x,y)$, $(x,y)\in\tilde T(\tilde S_1)$, we have $D\tilde T^{-1}_{(x,y)}(\bar v)\in K^s(\tilde T^{-1}(x,y))$, and if $D\tilde T^{-1}_{(x,y)}(\bar v)=\bar w\equiv(w_1,w_2)$ then $|w_2|\ge 0.01C_0^{-4}h^{-1-\nu}|v_2|$ and $|\bar w|\le 25C_0^4h^{-1-\nu}|\bar v|$.

Before to begin the proof of Lemma 13 we will formulate and proof two extra lemmas that give estimates of the angle between images of vectors under linear maps.

Lemma 14. *For any two vectors $\bar{u}_1, \bar{u}_2$ and any linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ the following inequality holds:*

$$\sin\angle(A\bar{u}_1, A\bar{u}_2) \le \|A\| \cdot \|A^{-1}\| \cdot |\sin\angle(\bar{u}_1, \bar{u}_2)|$$

Proof of Lemma 14. Take two vectors $\bar{s}_1$ and $\bar{s}_2$ such that $\bar{s}_2 \perp (\bar{s}_1 - \bar{s}_2)$ and $\bar{s}_1 \| \bar{u}_1$, $\bar{s}_2 \| \bar{u}_2$. In this case $|\sin\angle(\bar{u}_1, \bar{u}_2)| = \frac{|\bar{s}_1 - \bar{s}_2|}{|\bar{s}_1|}$. Now we have

$$\sin\angle(A\bar{u}_1, A\bar{u}_2) \le \frac{|A\bar{s}_1 - A\bar{s}_2|}{|A\bar{s}_1|} \le \frac{\|A\||\bar{s}_1 - \bar{s}_2|}{\|A^{-1}\|^{-1}|\bar{s}_1|} = \|A\| \cdot \|A^{-1}\| \cdot |\sin\angle(\bar{u}_1, \bar{u}_2)|$$

Lemma 14 is proved.

Lemma 15. *For any vector $\bar{u} \in \mathbb{R}^2$, $\bar{u} \ne 0$, and any linear maps $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ the following inequality holds:*

$$\sin\angle(A\bar{u}, B\bar{u}) \le \|A\| \cdot \|A - B\|.$$

Proof of Lemma 15.

$$\sin\angle(A\bar{u}, B\bar{u}) \le \frac{|A\bar{u} - B\bar{u}|}{|A\bar{u}|} \le \frac{\|A - B\|}{\|A\|^{-1}} = \|A\| \cdot \|A - B\|.$$

Lemma 15 is proved.

Proof of Lemma 13. We will prove the first part of the statement. The proof of the second part is completely the same.

Take a vector $\bar{v} \equiv (v_1, v_2) \in K^u(x, y)$, $(x, y) \in \tilde{S}_1$.

Consider the following points:

$$P_1 = \rho^{-1}(x, y) \in S, \quad P_2 = L^n(P_1) \in R^u, \quad P_3 = \Psi_h^{-1} \circ C_h^{-1}(P_2) \in \Pi_{r_0, E_0},$$

$$P_4 = H^{k(h)}(P_3) \in \Pi_{r_0, E_0}, \quad P_5 = C_h \circ \Psi_h(P_4) \in R^s, \quad P_6 = L^n(P_5) \in T(S_1) \subset S,$$

and denote by (x_i, y_i) the coordinates of the point P_i, $i = 1, \dots, 6$. We will follow the image of the vector along this sequence of points and estimate the angle between that image and coordinate axes and the size of the image.

Step 1. The vector $\bar{v}^{(1)} = D\rho^{-1}(\bar{v}) \in T_{P_1}U$ has coordinates (lv_1, lv_2) and $\bar{v}^{(1)} \in K^u(P_1)$. Notice that $|\bar{v}^{(1)}| = l|\bar{v}|$.

Step 2. By Lemma 11 the vector $\bar{v}^{(2)} = DN_h^n(\bar{v}^{(1)})$ has coordinates

$$(\lambda^n lv_1 + O(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|)l|\bar{v}|, \lambda^{-n} lv_2 + O((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|)l|\bar{v}|).$$

Since $|v_1| > 0.01C_0^{-6}h^{-1-h}|v_2|$, we have $|v_1| \leq |\bar{v}| \leq |v_1| + |v_2| \leq (1 + 100C_0^6 h^{1+\nu})|v_1|$, and hence

$$\frac{1}{4}l\lambda^n|\bar{v}| < \frac{l\lambda^n|\bar{v}|}{2(1+100C_0^6h^{1+\nu})} < \frac{1}{2}\lambda^n l|v_1| \leq |\bar{v}^{(2)}| \leq 2\lambda^n|\bar{v}^{(1)}| = 2\lambda^n l|\bar{v}|.$$

The angle between $\bar{v}^{(2)}$ and the vector $\bar{e}_1 \equiv (1,0)$ (we will assume that this angle is acute, otherwise consider the vector $(-1,0)$ instead) is not greater than $\frac{|v_2^{(2)}|}{|v_1^{(2)}|} \leq 4\lambda^{-2n}\frac{|v_2|}{|v_1|} < 400C_0^6h^{1+\nu}\lambda^2\mu(h)h^{1+\nu} = (400C_0^6\lambda^2)\mu(h)h^{2+2\nu}$.

Step 3. Set $\bar{v}^{(3)} = D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{v}^{(2)}$. Then $C_0^{-1}|\bar{v}^{(2)}| \leq |\bar{v}^{(3)}| \leq C_0|\bar{v}^{(2)}|$. Let us estimate the angle between $\bar{v}^{(3)}$ and the vector $\bar{e}_1 = (1,0)$. Let P^* be a projection of the point P_2 to the line $\{y=0\}$. Then $\mathrm{dist}(P_2, P^*) \leq l\lambda^{-n} = \lambda^{-2n+\frac{1}{10}}$. Since the image of the line $\{y=0\}$ under the map $\Psi_h^{-1} \circ C_h^{-1}$ is a line $\{E=0\}$, the image of the vector $\bar{e}_1 = (1,0)$ under the differential $D(\Psi_h^{-1} \circ C_h^{-1})$ has the form $(s,0) = s\bar{e}_1$. Now we have

$$\angle(\bar{v}^{(3)}, \bar{e}_1) = \angle(\bar{v}^{(3)}, s\bar{e}_1) = \angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{v}^{(2)}, D_{P^*}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right) \leq$$
$$\leq \angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{v}^{(2)}, D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right) + \angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1, D_{P^*}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right)$$

Now let us estimate each of the summands. Since all the angles that we consider are small, we can always assume that $\alpha < 2\sin\alpha < 2\alpha$ for all angles α that we consider. Due to Lemma 14 we have

$$\begin{aligned} (22)\quad & \angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{v}^{(2)}, D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right) \leq \\ & \leq 2\sin\angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{v}^{(2)}, D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right) \leq \\ & \leq 2C_0^2|\sin\angle(\bar{v}^{(2)}, \bar{e}_1)| \leq 2C_0^2 \cdot (400C_0^6\lambda^2)\mu(h)h^{2+2\nu} = 800C_0^8\lambda^2\mu(h)h^{2+2\nu} \end{aligned}$$

Due to Lemma 15 we have

$$\begin{aligned} (23)\quad & \angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1, D_{P^*}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right) \leq \\ & \leq 2\sin\angle\left(D_{P_2}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1, D_{P^*}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_1\right) \leq \\ & \leq 2C_0 \cdot C_0\,\mathrm{dist}(P_2, P^*) \leq 4C_0^2\lambda^{-2n+\frac{1}{10}} \leq 4C_0^2\lambda^{2.1}\mu(h)h^{1+\nu} \end{aligned}$$

Finally (if h is small enough and $\lambda = e^h$ is close to 1) we have

$$(24)\qquad \angle(\bar{v}^{(3)}, \bar{e}_1) \leq 800C_0^8\lambda^2\mu(h)h^{2+2\nu} + 4C_0^2\lambda^{2.1}\mu(h)h^{1+\nu} \leq 5C_0^2\mu(h)h^{1+\nu}$$

Step 4. Since $H(t,E) = (t+h, E)$, the estimates for $\bar{v}^{(3)}$ work for $\bar{v}^{(4)} = DH^{k(h)}(\bar{v}^{(3)})$ also.

Step 5. Consider $\bar{v}^{(5)} = D_{P_4}(C_h \circ \Psi_h)\bar{v}^{(4)} \in T_{P_5}U$. Notice that

$$(25)\qquad |\bar{v}^{(5)}| \geq C_0^{-1}|\bar{v}^{(4)}| \geq C_0^{-2}|\bar{v}^{(2)}| > \frac{1}{4}\,C_0^{-2}l\lambda^n|\bar{v}| \quad \text{and}$$

$$|\bar{v}^{(5)}| \le C_0|\bar{v}^{(4)}| \le C_0^2|\bar{v}^{(2)}| \le 2C_0^2 l\lambda^n|\bar{v}|. \tag{26}$$

Now let us estimate the angle between $\bar{v}^{(5)}$ and the ax Oy. Let $P^{\#}$ be a projection of the point P_5 on the line $\{x=0\}$. Take the vector $\bar{e}_2 = (0,1) \in T_{P^{\#}}U$ and consider the image $D_{P^{\#}}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_2 \in T_{\Psi_h^{-1}\circ C_h^{-1}(P^{\#})}\Pi_{r_0,E_0}$. The vector $D_{P^{\#}}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_2$ is tangent to the graph of the function $\Theta(t)$, and due to (9)

$$\frac{1}{2}\mu(h) < \angle(D_{P^{\#}}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_2, \bar{e}_1) < 2\mu(h). \tag{27}$$

From (24) we have

$$\frac{1}{5}\mu(h) < \angle(D_{P^{\#}}(\Psi_h^{-1} \circ C_h^{-1})\bar{e}_2, \bar{v}^{(4)}) < 5\mu(h).$$

Notice that $\operatorname{dist}(\Psi_h^{-1} \circ C_h^{-1}(P^{\#}), P_4) \le 2C_0\lambda^{-2n+\frac{1}{10}} \le 4C_0\mu(h)h^{1+\nu}$. This implies (in the way similar to Step 3) that for small enough h

$$\angle(\bar{v}^{(5)}, \bar{e}_2) < 5C_0^2\mu(h) + C_0 \cdot C_0 \cdot 4C_0\mu(h)h^{1+\nu} < 6C_0^2\mu(h), \tag{28}$$

$$\angle(\bar{v}^{(5)}, \bar{e}_2) > \frac{1}{5}\mu(h)C_0^{-2} - 4C_0^3\mu(h)h^{1+\nu} > \frac{1}{6}C_0^{-2}\mu(h). \tag{29}$$

Step 6. Set $\bar{v}^{(6)} = DN_h^n\bar{v}^{(5)} \in T_{P_6}U$. Let us estimate $|\bar{v}^{(6)}|$ first. If $\bar{v}^{(6)} = (v_1^{(6)}, v_2^{(6)})$ then by Lemma 11

$v_1^{(6)} = \left(\lambda^n + O\left(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|\right)\right)v_1^{(5)} + O\left((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|\right)v_2^{(5)}$ and

$v_2^{(6)} = O\left(\sqrt{\mu(h)h^{1+\nu}}|\log(\mu(h)h^{1+\nu})|\right)v_1^{(5)} + \left(\lambda^{-n} + O\left((\mu(h)h^{1+\nu})^{\frac{3}{2}}|\log(\mu(h)h^{1+\nu})|\right)\right)v_2^{(5)}$.

From (29) we have

$$|v_1^{(5)}| \ge \frac{1}{12}C_0^{-2}\mu(h)|\bar{v}^{(5)}| > \frac{1}{48}C_0^{-4}l\lambda^n|\bar{v}|\mu(h) \quad \text{and} \quad |v_2^{(5)}| \le |\bar{v}^{(5)}| \tag{30}$$

Therefore

$$|v_1^{(6)}| \ge \frac{1}{2}\lambda^n|v_1^{(5)}| \ge \frac{1}{96}C_0^{-4}l\lambda^{2n}\mu(h)|\bar{v}|, \quad |v_2^{(6)}| \le 2\lambda^{-n}|\bar{v}^{(5)}| \le 4C_0^2 l|\bar{v}|.$$

Now let us show that $\bar{v}^{(6)} \in K^u(P_6)$. Indeed, if h is small enough (and $\lambda = e^h$ is close to 1) then $\frac{|v_1^{(6)}|}{|v_2^{(6)}|} \ge \frac{1}{2\cdot 96}C_0^{-6}\lambda^{-2}h^{-1-\nu} > 0.001C_0^{-6}h^{-1-\nu}$. Therefore $\bar{v}^{(6)} \in K^u(P_6)$, and $D\rho(\bar{v}^{(6)}) = D\tilde{T}(\bar{v}) \in K^u(\rho(P_6))$. Also if $\bar{w} = (w_1, w_2) = D\tilde{T}(\bar{v})$ then

$$|w_1| = l^{-1}|v_1^{(6)}| > \frac{1}{96}\lambda^{-2}C_0^{-4}h^{-1-\nu}|\bar{v}| > 0.01C_0^{-4}h^{-1-\nu}|v_1|$$

At the same time we have

$$(31)\quad |D\tilde{T}(\bar{v})| = l^{-1}|\bar{v}^{(6)}| \le l^{-1}(|v_1^{(6)}| + |v_2^{(6)}|) \le l^{-1}(2\lambda^n|v_1^{(5)}| + 4C_0^2 l|\bar{v}|) \le$$
$$\le l^{-1}(2\lambda^n|\bar{v}^{(5)}|\sin\angle(\bar{v}^{(5)}, \bar{e}_2) + 4C_0^2 l|\bar{v}|) \le l^{-1}(2\lambda^n \cdot 2C_0^2 l\lambda^n|\bar{v}| \cdot 6C_0^2\mu(h) + 4C_0^2 l|\bar{v}|) \le$$
$$\le 24C_0^4\lambda^{2n}\mu(h)|\bar{v}| + 4C_0^2|\bar{v}| \le 25C_0^4 h^{-1-\nu}|\bar{v}|.$$

Lemma 13 is proved.

0.15. **Markov partition and its thickness.**

Standard arguments of the hyperbolic theory (see, for example, [IL]) show that the Cone condition (Lemmas 12 and 13) together with the geometry of the map $\tilde{T}$ imply the existence of the hyperbolic fixed point $\mathbf{Q}$ of the map $\tilde{T}$ in $\tilde{S}_1 \cap \tilde{T}(\tilde{S}_1)$. Our choice of the homoclinic points q_h^u and q_h^s implies that the eigenvalues of $\mathbf{Q}$ are posilive. Denote the heteroclinic point where $W_{loc}^s(\mathbf{Q})$ intersects $W^u(O) = \{(0,x)|x \in \mathbb{R}\}$ by $(x_s, 0)$, and the heteroclinic point where $W_{loc}^u(\mathbf{Q})$ intersects $W^s(O) = \{(y,0)|y \in \mathbb{R}\}$ by $(0, y_u)$.

Denote the segments of stable and unstable manifolds that connect the fixed points O and $\mathbf{Q}$ with these heteroclinic points by

$\gamma^u(O)$ – connects O and $(x_s, 0)$, $\quad\gamma^s(O)$ – connects O and $(0, y_u)$;

$\gamma^u(\mathbf{Q})$ – connects $\mathbf{Q}$ and $(0, y_u)$, $\quad\gamma^s(\mathbf{Q})$ – connects $\mathbf{Q}$ and $(x_s, 0)$.

Notice that $\gamma^s(\mathbf{Q}) \subset \tilde{S}_1$ and $\gamma^u(\mathbf{Q}) \subset \tilde{T}(\tilde{S}_1)$.

Let $\mathbf{S}$ be the square formed by $\gamma^u(O)$, $\gamma^s(O)$, $\gamma^u(\mathbf{Q})$ and $\gamma^s(\mathbf{Q})$, $\mathbf{S} \subset \tilde{S}$.

Now define $\mathbf{S}_0 = \rho(\rho^{-1}(\mathbf{S}) \cap N_h \circ \rho^{-1}(\mathbf{S})) \subset \tilde{S}_0$ and $\mathbf{S}_1 = \rho(S_1 \cap \rho^{-1}(\mathbf{S})) \subset \tilde{S}_1$, see Fig ?. Notice that one of the vertical edges of $\mathbf{S}_1$ is $\gamma^s(\mathbf{Q})$ and another is an intersection of $\mathbf{S}$ and a vertical edge of $\tilde{S}_1$, and therefore it intersects $W^u(O)$ at the point $\rho(N_h^{-n}(1,0)) = \rho(\lambda^{-n}) = \lambda^{n-\frac{1}{10}} \cdot \lambda^{-n} = \lambda^{-\frac{1}{10}}$. Similarly, $\tilde{T}(\mathbf{S}_1)$ has a vertical edge $[\lambda^{-\frac{1}{10}}, y_u] \subset Oy$.

Define now $\mathbf{T} = \tilde{T}|_{\mathbf{S}}$. The maximal invariant set of $\mathbf{T}$ in $\mathbf{S}$, $\Lambda = \cap_{n\in\mathbb{Z}}\mathbf{T}^{-n}(\mathbf{S})$, is a "horse-shoe"-type basic set with Markov partition $\mathcal{P} = \{\mathbf{S}_0, \mathbf{S}_1\}$. The map $\mathbf{T} : \mathbf{S}_0 \cup \mathbf{S}_1 \to \mathbf{S}$ belongs to class $\mathcal{F}$ (see definition 5).

Consider now the Markov partitions

$$\mathcal{P}^s = \{[0, \lambda^{-1}x_s], [\lambda^{-\frac{1}{10}}, x_s]\} \quad \text{and} \quad \mathcal{P}^u = \{[0, \lambda^{-1}y_u], [\lambda^{-\frac{1}{10}}, y_u]\}$$

of the Cantor sets $K^s \subset Ox$ and $K^u \subset Oy$ associated with the hyperbolic set Λ. We have

$$\tau_L(\mathcal{P}^s) = \frac{\lambda^{-1}x_s}{\lambda^{-\frac{1}{10}} - \lambda^{-1}x_s}, \quad \tau_R(\mathcal{P}^s) = \frac{x_s - \lambda^{-\frac{1}{10}}}{\lambda^{-\frac{1}{10}} - \lambda^{-1}x_s},$$
$$\tau_L(\mathcal{P}^u) = \frac{\lambda^{-1}y_u}{\lambda^{-\frac{1}{10}} - \lambda^{-1}y_u}, \quad \tau_R(\mathcal{P}^u) = \frac{y_u - \lambda^{-\frac{1}{10}}}{\lambda^{-\frac{1}{10}} - \lambda^{-1}y_u}.$$

Lemma 16. *The following estimates hold for all $h \in (0, h_0)$ if h_0 is small enough:*

$$\frac{1}{2}h^{-1} \le \tau_L(\mathcal{P}^s) \le \frac{10}{9}h^{-1}, \quad 0.01C_0^{-4}h^\nu \le \tau_R(\mathcal{P}^s) \le 900C_0^4h^\nu,$$
$$\frac{1}{2}h^{-1} \le \tau_L(\mathcal{P}^u) \le \frac{10}{9}h^{-1}, \quad 0.01C_0^{-4}h^\nu \le \tau_R(\mathcal{P}^u) \le 900C_0^4h^\nu.$$

Proof of Lemma 16. We will prove only estimates for the partition $\mathcal{P}^s$ (for $\mathcal{P}^u$ everything is the same). Notice first that $\lambda^{-\frac{1}{10}} - \lambda^{-1}x_s \le x_s(1 - \lambda^{-1})$. Therefore

$$\tau_L(\mathcal{P}^s) = \frac{\lambda^{-1}x_s}{\lambda^{-\frac{1}{10}} - \lambda^{-1}x_s} \ge \frac{\lambda^{-1}x_s}{x_s - \lambda^{-1}x_s} = \frac{\lambda^{-1}}{1-\lambda^{-1}} = \frac{1}{\lambda - 1} = \frac{1}{e^h - 1} \ge \frac{1}{2}h^{-1},$$

since $e^h - 1 \le 2h$ for small h.

On the other hand, since $x_s \le 1$

$$\tau_L(\mathcal{P}^s) = \frac{\lambda^{-1}x_s}{\lambda^{-\frac{1}{10}} - \lambda^{-1}x_s} \le \frac{\lambda^{-1}}{\lambda^{-\frac{1}{10}} - \lambda^{-1}} = \frac{1}{\lambda^{\frac{9}{10}} - 1} = \frac{1}{e^{\frac{9}{10}h} - 1} \le \frac{10}{9}h^{-1}$$

Now let us estimate $\tau_R(\mathcal{P}^s)$. Denote by I the segment of $W^u_{loc}(O)$ between the points $(\lambda^{-\frac{1}{10}}, 0)$ and $(x_s, 0)$ (i.e. the bottom horizontal edge of $\mathbf{S}_1$), *length*$(I) = x_s - \lambda^{-\frac{1}{10}}$. Due to Lemma 13

$$0.01C_0^{-4}h^{-1-\nu}\mathit{length}(I) \le \mathit{length}(\mathbf{T}(I)) \le 25C_0^4h^{-1-\nu}\mathit{length}(I).$$

Since $\frac{1}{2} \le \mathit{length}(\mathbf{T}(I)) \le 2$ (this follows from the Cone condition again), we have

$$\frac{1}{50}C_0^{-4}h^{1+\nu} \le \mathit{length}(I) \le 200C_0^4h^{1+\nu}$$

Hence

$$\tau_R(\mathcal{P}^s) \ge \frac{\frac{1}{50}C_0^{-4}h^{1+\nu}}{x_s(1-\lambda^{-1})} \ge \frac{\frac{1}{50}C_0^{-4}h^{1+\nu}}{1 - e^{-h}} \ge 0.01C_0^{-4}h^\nu$$

since $1 - e^{-h} \le h$ for $h \ge 0$, and

$$\tau_R(\mathcal{P}^s) \le \frac{200C_0^4h^{1+\nu}}{\lambda^{-\frac{1}{10}} - \lambda^{-1}} \le \frac{270C_0^4h^{1+\nu}}{\lambda^{\frac{9}{10}} - 1} = \frac{270C_0^4h^{1+\nu}}{e^{\frac{9}{10}h} - 1} \le 900C_0^4h^\nu.$$

Lemma 16 is proved.

Lemma 17. *If $h_0 > 0$ is small enough then for all $h \in (0, h_0)$ we have*

$$\mathrm{dist}(\mathbf{S}_0, \mathbf{S}_1) \geq 0.1h \quad \text{and} \quad \mathrm{dist}(\mathbf{T}(\mathbf{S}_0), \mathbf{T}(\mathbf{S}_1)) \geq 0.1h.$$

Proof of Lemma 17. We will prove the first inequality only, since the proof of the second one is completely similar.

Notice that the vertical boundaries of $\mathbf{S}_0$ and $\mathbf{S}_1$ are tangent to the cone field $\{K^u\}$. Consider the left vertical edge of $\mathbf{S}_1$ and the right vertical edge of $\mathbf{S}_0$. Their lowest points are $(\lambda^{-\frac{1}{10}}, 0)$ and $(\lambda^{-1}x_s, 0)$, and the distance between them is equal to

$$\lambda^{\frac{1}{10}} - \lambda^{-1}x_s \geq \lambda^{\frac{1}{10}} - \lambda^{-1} = \lambda^{-1}(\lambda^{\frac{9}{10}} - 1) = e^{-h}(e^{\frac{9}{10}h} - 1) \geq e^{-h} \cdot \frac{9}{10}h \geq \frac{9}{20}h$$

if $h \in (0, h_0)$ and h_0 is small enough. From the cone condition we have that the difference between x-coordinates of any two points on those edges is greater than

$$\frac{9}{20}h - 2 \cdot 1000C_0^6 h^{1+\nu} \geq \frac{9}{20}h\left(1 - \frac{40000}{9}C_0^6 h^\nu\right) \geq \frac{9}{40}h > 0.1h$$

if h is small enough.

Lemma 17 is proved.

0.16. **Estimates of derivatives: verification of the conditions of Distortion Theorem.**

We proved that the map $\mathbf{T} : \mathbf{S}_0 \cup \mathbf{S}_1 \to \mathbf{S}$ has an invariant locally maximal hyperbolic set Λ which is a two-component Smale horseshoe (i.e. $\mathbf{T}$ belongs to the class $\mathcal{F}$) and obtained estimates of the lateral thicknesses of the corresponding Markov partitions. In order to get estimates of the lateral thicknesses of the related Cantor sets we need to estimate the distortion of the corresponding mappings.

Denote the differential of the map $\mathbf{T} : \mathbf{S}_0 \cup \mathbf{S}_1 \to \mathbf{S}$ by $D\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are smooth functions over $\mathbf{S}_0 \cup \mathbf{S}_1$. Then the differential of the inverse map $\mathbf{T}^{-1} : \mathbf{T}(\mathbf{S}_0) \cup \mathbf{T}(\mathbf{S}_1) \to \mathbf{S}$ has the form $D\mathbf{T}^{-1} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}$, where $\tilde{a} = a \circ \mathbf{T}^{-1}$, $\tilde{b} = b \circ \mathbf{T}^{-1}$, $\tilde{c} = c \circ \mathbf{T}^{-1}$ and $\tilde{d} = d \circ \mathbf{T}^{-1}$. Notice that this notation agrees with the notation of Definition 6.

Lemma 18. *Consider the restriction of the map $\mathbf{T}$ to the rectangle $\mathbf{S}_1$. There exists a constant $C_1 > 1$ (independent of h) such that*

(1) $|d| \leq \lambda^{-2n}C_1$, $|b|, |c| \leq C_1$,

(2) $C_1^{-1}h^{-1-\nu} \leq |a| \leq C_1 h^{-1-\nu}$,

(3) $\left|\frac{\partial b}{\partial x}\right|, \left|\frac{\partial b}{\partial y}\right|, \left|\frac{\partial c}{\partial x}\right|, \left|\frac{\partial c}{\partial y}\right|, \left|\frac{\partial a}{\partial y}\right| \leq C_1,$

(4) $\left|\frac{\partial d}{\partial x}\right| \leq \lambda^{-2n} C_1, \ \left|\frac{\partial d}{\partial y}\right| \leq \lambda^{-4n} C_1,$

(5) $\left|\frac{\partial a}{\partial x}\right| \leq C_1 h^{-2-\nu},$

(6) $\left|\frac{\partial \tilde{b}}{\partial x}\right|, \left|\frac{\partial \tilde{b}}{\partial y}\right|, \left|\frac{\partial \tilde{c}}{\partial x}\right|, \left|\frac{\partial \tilde{c}}{\partial y}\right|, \left|\frac{\partial \tilde{a}}{\partial x}\right| \leq C_1,$

(7) $\left|\frac{\partial \tilde{d}}{\partial y}\right| \leq \lambda^{-2n} C_1, \ \left|\frac{\partial \tilde{d}}{\partial x}\right| \leq \lambda^{-4n} C_1,$

(8) $\left|\frac{\partial \tilde{a}}{\partial y}\right| \leq C_1 h^{-2-\nu}.$

Proof of Lemma 18. We denoted

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = D(C_h \circ \Psi_h \circ H^{h(h)} \circ \Psi_h^{-1} \circ C_h^{-1}),$$

therefore

$$D\mathbf{T}|_{\mathbf{S}_1}(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix} = \begin{pmatrix} \lambda^{2n} a_0(\lambda^{\frac{1}{10}} x, \lambda^{-2n+\frac{1}{10}} y) & b_0(\lambda^{\frac{1}{10}} x, \lambda^{-2n+\frac{1}{10}} y) \\ c_0(\lambda^{\frac{1}{10}} x, \lambda^{-2n+\frac{1}{10}} y) & \lambda^{-2n} d_0(\lambda^{\frac{1}{10}} x, \lambda^{-2n+\frac{1}{10}} y) \end{pmatrix}.$$

The C^3-norm of the map $C_h \circ \Psi_h \circ H^{h(h)} \circ \Psi_h^{-1} \circ C_h^{-1}$ (as well as of its inverse) is uniformly bounded by some constant independent of h. Now all the inequalities (1), (3), (4) follow for C_1 large enough. Inequalities (2) follow from Lemma 13, we just need to assume that $C_1 > 100C_0^4$.

Now let us prove the inequality (5). Since $a(x,y) = \lambda^{2n} a_0(\lambda^{\frac{1}{10}} x, \lambda^{-2n+\frac{1}{10}} y)$, we have

$$\frac{\partial a}{\partial x}(x,y) = \lambda^{2n+\frac{1}{10}} \partial_1 a_0(\lambda^{\frac{1}{10}} x, \lambda^{-2n+\frac{1}{10}} y) = \lambda^{2n+\frac{1}{10}} \partial_1 a_0(\lambda^{\frac{1}{10}} x, 0) + O(1).$$

In order to distinguish the points from R^u and from R^s let us denote the coordinates in R^u by $(\mathbf{x}, \mathbf{y})$ and the coordinates in R^s by $(\mathbf{X}, \mathbf{Y})$. Then the map $C_h \circ \Psi_h \circ H^{k(h)} \circ \Psi_h^{-1} \circ C_h^{-1} : R^u \to U$ can be represented as a composition

$$(\mathbf{x}, \mathbf{y}) \mapsto (t(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) \mapsto (\mathbf{X}(t, E), \mathbf{Y}(t, E)), \quad (t, E) \in \Pi_{r_0, E_0},$$

where

$$(t(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) = H^{k(h)} \circ \Psi_h^{-1} \circ C_h^{-1}(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad (\mathbf{X}(t, E), \mathbf{Y}(t, E)) = C_h \circ \Psi_h(t, E).$$

We have

$$a_0(\mathbf{x}, \mathbf{y}) = \frac{\partial}{\partial \mathbf{x}} \mathbf{X}(t(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) \quad \text{and} \quad \partial_1 a_0 \equiv \frac{\partial a_0}{\partial \mathbf{x}} = \frac{\partial^2}{\partial \mathbf{x}^2} \mathbf{X}(t(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})).$$

In particular, since $E(\mathbf{x},0)=0$,

$$(32)\quad \partial_1 a_0(\mathbf{x},0)=\frac{d^2}{d\mathbf{x}^2}\mathbf{X}(t(\mathbf{x},0),0)=\frac{d}{d\mathbf{x}}\left(\frac{\partial \mathbf{X}}{\partial t}(t(\mathbf{x},0),0)\cdot\frac{\partial t}{\partial \mathbf{x}}(\mathbf{x},0)\right)=$$
$$=\frac{\partial^2 \mathbf{X}}{\partial t^2}(t(\mathbf{x},0),0)\cdot\left(\frac{\partial t}{\partial \mathbf{x}}(\mathbf{x},0)\right)^2+\frac{\partial \mathbf{X}}{\partial t}(t(\mathbf{x},0),0)\cdot\frac{\partial^2 t}{\partial \mathbf{x}^2}(\mathbf{x},0)=$$
$$=\frac{\partial^2 \mathbf{X}}{\partial t^2}(t(\mathbf{x},0),0)\cdot\left(\frac{\partial t}{\partial \mathbf{x}}(\mathbf{x},0)\right)^2+a_0(\mathbf{x},0)\cdot\left(\frac{\partial t}{\partial \mathbf{x}}(\mathbf{x},0)\right)^{-1}\cdot\frac{\partial^2 t}{\partial \mathbf{x}^2}(\mathbf{x},0)$$

From the Cone condition (more precisely, from Steps 3-5 of the proof of Lemma 13) we know that $|a_0(\mathbf{x},0)|\le 7C_0^4\mu(h)$. Also since C^3-norms of maps $C_h\circ\Psi_h$ and $H^{k(h)}\circ\Psi_h^{-1}\circ C_h^{-1}$ are bounded by C_0, we have

$$|\partial a_0(\mathbf{x},0)|\le\left|\frac{\partial^2 \mathbf{X}}{\partial t^2}(t(\mathbf{x},0),0)\right|\cdot C_0^2+7C_0^4\mu(h)\cdot C_0^2$$

Now we need to estimate $\left|\frac{\partial^2 \mathbf{X}}{\partial t^2}(t(\mathbf{x},0),0)\right|$. Notice that the image of the Oy ax under the map $C_h\circ\Psi_h$ is a graph of the function $E=\Theta^*(t)$, and therefore $\mathbf{X}(t,\Theta^*(t))=0$. This implies that

$$\frac{d}{dt}(\mathbf{X}(t,\Theta^*(t))=0=\frac{\partial \mathbf{X}}{\partial t}(t,\Theta^*(t))+\frac{\partial \mathbf{X}}{\partial E}(t,\Theta^*(t))\cdot\dot{\Theta}^*(t)$$

and

$$(33)\quad \frac{d^2}{dt^2}(\mathbf{X}(t,\Theta^*(t))=0=$$
$$=\frac{\partial^2 \mathbf{X}}{\partial t^2}(t,\Theta^*(t))+2\frac{\partial^2 \mathbf{X}}{\partial t\partial E}(t,\Theta^*(t))\dot{\Theta}^*(t)+\frac{\partial^2 \mathbf{X}}{\partial E^2}(t,\Theta^*(t))(\dot{\Theta}^*(t))^2+\frac{\partial \mathbf{X}}{\partial E}(t,\Theta^*(t))\ddot{\Theta}^*(t)$$

Since for $\Theta^*(t)$, $\dot{\Theta}^*(t)$ and $\ddot{\Theta}^*(t)$ we have asymptotics (9) (see Proposition 9 and Remark 13), we get

$$\left|\frac{\partial^2 \mathbf{X}}{\partial t^2}(t,\Theta^*(t))\right|\le 2C_0\cdot 2\mu(h)+C_0(2\mu(h))^2+C_0\cdot 4\pi h^{-1}\mu(h)<20C_0h^{-1}\mu(h)$$

if h is small enough.

At the same time by the mean value theorem we have

$$\left|\frac{\partial^2 \mathbf{X}}{\partial t^2}(t,0)\right|\le\left|\frac{\partial^2 \mathbf{X}}{\partial t^2}(t,\Theta^*(t))\right|+C_0|\Theta^*(t)|<20C_0h^{-1}\mu(h)+C_0\frac{1}{\pi}h\mu(h)\le 40C_0h^{-1}\mu(h)$$

Finally we have

$$|\partial_1 a_0(\mathbf{x},0)|\le 40C_0h^{-1}\mu(h)\cdot C_0^2+7C_0^4\mu(h)\cdot C_0^2\le 50C_0^3h^{-1}\mu(h)$$

and (setting $\mathbf{x} = \lambda^{\frac{1}{10}} x$)

$$(34)\quad |a(x,y)| \le \lambda^{2n+\frac{1}{10}} \cdot 50C_0^3 h^{-1}\mu(h) + O(1) \le$$
$$\le (\mu(h))^{-1} h^{-1-\nu} \lambda^{\frac{1}{10}} \cdot 50C_0^3 h^{-1}\mu(h) + O(1) \le C_1 h^{-2-\nu}$$

for large C_1 (independent of $h \in (0, h_0)$).

Properties (6) – (8) are symmetric to the properties (3) – (5). Lemma 18 is proved.

Lemma 19. *The variation of* $\log|a(x,y)|$ *in* $\mathbf{S}_1$ *is less than* $600C_1^2C_0^6h^\nu$.

Proof of Lemma 19. Take two points (x_1, y_1) and (x_2, y_2) from $\mathbf{S}_1$. We want to estimate $|\log a(x_1,y_1) - \log a(x_2,y_2)|$ by using the mean value theorem. Generally speaking, the set $\mathbf{S}_1$ is not convex, so we need some preparations to apply it.

Let $\tilde\gamma$ be the intersection $\tilde\gamma = \mathbf{T}(\mathbf{S})_1 \cap \{x = \frac{1}{2}\}$. Then $\hat\gamma = \mathbf{T}^{-1}(\tilde\gamma)$ is a smooth curve tangent to the cone field $\{K^s\}$, $\hat\gamma \subset \mathbf{S}_1$. Denote $\hat x_1 = \{y = y_1\} \cap \hat\gamma$ and $\hat x_2 = \{y = y_2\} \cap \hat\gamma$. Notice that the whole interval with the end points (x_1, y_1) and $(\hat x_1, y_1)$ belongs to $\mathbf{S}_1$, as well as the interval with end points (x_2, y_2) and $(\hat x_2, y_2)$. Now we have

$$(35)\quad |\log a(x_1,y_1) - \log a(x_2,y_2)| \le$$
$$\le |\log a(x_1,y_1) - \log a(\hat x_1,y_1)| + |\log a(\hat x_1,y_1) - \log a(\hat x_2,y_2)| + |\log a(\hat x_2,y_2) - \log a(x_2,y_2)|$$

Due to the Cone condition the width of $\mathbf{S}_1$ is not greater than $200C_0^4h^{1+\nu}$. By the mean value theorem we have

$$(36)\quad |\log a(x_1,y_1) - \log a(\hat x_1,y_1)| \le$$
$$\le \frac{1}{|a(x_1^*,y_1)|}\left|\frac{\partial a}{\partial x}(x_1^*,y_1)\right| |x_1 - \hat x_1| \le C_1h^{1+\nu} \cdot C_1h^{-2-\nu} \cdot 200C_0^4h^{1+\nu} = 200C_1^2C_0^4h^\nu$$

Similarly

$$|\log a(\hat x_2,y_2) - \log a(x_2,y_2)| \le 200C_1^2C_0^4h^\nu$$

Now parameterize the curve $\hat\gamma$ by the parameter y, $\hat\gamma = \hat\gamma(x(y), y), y \in [y_1, y_2]$ (or $y \in [y_2, y_1]$ if $y_2 < y_1$). Consider a function $g(y) = \log a(\hat\gamma(x(y), y)$. Since $\hat\gamma$ is tangent to the cone field $\{K^s\}$, for some $y^* \in [y_1, y_2]$ we have

(37)
$$|g(y_1) - g(y_2)| = |g'(y^*)| \cdot |y_1 - y_2| = \frac{1}{|a(\hat\gamma(x(y^*), y^*)|} \cdot \left|\frac{\partial a}{\partial x}\hat\gamma'_x + \frac{\partial a}{\partial y}\hat\gamma'_y\right| \cdot |y_1 - y_2| \le$$
$$\le C_1h^{1+\nu}\left(\left|\frac{\partial a}{\partial x}\right| |\hat\gamma'_x| + \left|\frac{\partial a}{\partial y}\right| |\hat\gamma'_y|\right) \le C_1h^{1+\nu}(C_1h^{-2-\nu} \cdot 100C_0^6h^{1+\nu} + C_1) \le$$
$$\le C_1h^\nu(100C_1C_0^6 + C_1h) \le 200C_1^2C_0^6h^\nu$$

Finally we have

$$|\log a(x_1, y_1) - \log a(x_2, y_2)| \leq 400C_1^2 C_0^4 h^\nu + 200C_1^2 C_0^6 h^\nu < 600C_1^2 C_0^6 h^\nu$$

Lemma 19 is proved.

The following Proposition directly follows from Lemmas 12, 13, 17, 18 and 19.

Proposition 20. *The map* $\mathbf{T} : \mathbf{S}_1 \cup \mathbf{S}_2 \to \mathbf{S}$ *belongs to the class* $\mathcal{F}(C^*, \gamma, \varepsilon)$ *(see definition 6), where* $C^* = 120C_1^4 C_0^6$, $\gamma = 1200C_1^3 C_0^6 h^\nu$ *and* $\varepsilon = 120C_1^3 C_0^6 h^{1+\nu}$.

0.17. **Proof of the first of the Main Theorems.**

Properties 1. and 2. of Theorem 1 clearly follow from the construction and the Cone condition. Let us combine now Proposition 20 with Duarte Distortion Theorem (Theorem 3) and Lemma 16. Let us assume that h is small enough so that $e^{D(C^*, \varepsilon, \gamma)} < 2$. Then we have

$$\frac{1}{4}h^{-1} \leq \tau_L(K^s) \leq \frac{20}{9}h^{-1}, \quad \frac{1}{200}C_0^{-4}h^\nu \leq \tau_R(K^s) \leq 1800C_0^4 h^\nu,$$

$$\frac{1}{4}h^{-1} \leq \tau_L(K^u) \leq \frac{20}{9}h^{-1}, \quad \frac{1}{200}C_0^{-4}h^\nu \leq \tau_R(K^u) \leq 1800C_0^4 h^\nu.$$

Therefore

$$\tau_L(K^s)\tau_R(K^s) \geq \frac{1}{800}C_0^{-4}h^{-1+\nu} \to \infty \quad \text{as} \quad h \to 0 \quad (\text{i.e. } a \to -1).$$

Similarly $\tau_L(K^u)\tau_R(K^u) \to \infty$ as $a \to -1$, so $\tau_{LR}(\Lambda) \to \infty$ as $a \to -1$.

To check the property 4., we apply Proposition 5 (notice that we are exactly in the setting of Remark 6). We have

$$(38) \quad \dim_H K^s \geq \frac{\log\left(1 + \frac{\tau_L(K^s)}{1+\tau_R(K^s)}\right)}{\log\left(1 + \frac{1+\tau_L(K^s)}{\tau_R(K^s)}\right)} \geq \frac{\log\left(1 + \frac{\frac{1}{4}h^{-1}}{1+1800C_0^4 h^\nu}\right)}{\log\left(1 + \frac{1+\frac{20}{9}h^{-1}}{\frac{1}{200}C_0^{-4}h^\nu}\right)} =$$

$$= \frac{\log\left[h^{-1}\left(h + \frac{1}{4+7200C_0^4 h^\nu}\right)\right]}{\log\left[h^{-1-\nu}\left(h^{1+\nu} + 200C_0^4(\frac{20}{9} + h)\right)\right]} = \frac{1 - O(\log h^{-1})}{1 + \nu - O(\log h^{-1})} > \frac{1}{1+2\nu}$$

if h is small enough. Since ν could be chosen arbitrary small, $\dim_H K^s \to 1$ as $h \to 0$ (i.e. $a \to -1$). Similarly $\dim_H K^u \to 1$ as $a \to -1$. Since $\dim_H \Lambda = \dim_H K^s + \dim_H K^u$ ([MM], see also [PV]), we have

$$\dim_H \Lambda \to 2 \;\; \text{as } a \to -1.$$

Theorem 1 is proved.

0.18. **Sketch of the proof of the second of the Main Theorems.**

Step 1. Choose a sequence of parameters $\{\mu_n\}$, $\mu_n \to 0$, such that f_{μ_n} has a quadratic homoclinic tangency and a transversal homoclinic points.

Step 2. Using the renormalization technics by Mora-Romero [MR1] and Theorem 1 one can show that near each μ_n there is an open interval U_n such that for $\mu \in U_n$ the map f_μ has an invariant locally maximal transitive hyperbolic set Λ^*_μ with persistent homoclinic tangencies.

Step 3. The hyperbolic saddle P_μ and the set Λ^*_μ are homoclinically related, see Lemma 2 from [Du1]. Therefore for every $\mu \in U_n$ there exists a basic set Λ_μ such that $P_\mu \in \Lambda_\mu$ and $\Lambda^*_\mu \subset \Lambda_\mu$. Since Λ^*_μ has persistent homoclinic tangencies, so does Λ_μ. This proves the part (1).

Step 4. There is a dense subset $D_n \subset U_n$ such that for every $\mu \in D_n$ there is some homoclinic tangency for the fixed point P_μ. This is a consequence of a standard arguments, see [PT]. The part (2) is proved.

Step 5. Consider local invariant manifolds $W^u_\varepsilon(P_\mu)$ and $W^s_\varepsilon(P_\mu)$, and set

$$W^u_\mu(k) = f^k_\mu(W^u_\varepsilon(P_\mu)) \quad \text{and} \quad W^s_\mu(k) = f^{-k}_\mu(W^s_\varepsilon(P_\mu)).$$

We have

$$W^u(P_\mu) = \cup_{k\geq 0} W^u_\mu(k) \quad \text{and} \quad W^s(P_\mu) = \cup_{k\geq 0} W^s_\mu(k).$$

Since we consider a generic unfolding $\{f_\mu\}$, we can assume that for an open and dense set $O(k) \subset U_n$ we have $W^s_\mu(k) \pitchfork W^u_\mu(k)$. From [MR1] it follows that there exists an open and dense subset $V(k) \subset O(k) \subset U_n$ such that for every $\mu \in V(k)$ and every $x \in W^s_\mu(k) \cap W^u_\mu(k)$ there exists an elliptic periodic point p such that $\text{dist}(p, x) < \frac{1}{k}$. Therefore for every μ from a residual set $\mathcal{R}_1 = \cap_{k\geq 1} V(k)$ the homoclinic class $H(P_\mu, f_\mu)$ is accumulated by elliptic points, and this proves (3.1).

Step 6. From Theorem 1 and [MR1] it follows that for every $m \in \mathbb{N}$ there exists an open and dense subset $A(m) \subset U_n$ such that for every $\mu \in A(m)$ there exists a hyperbolic set $\Lambda^\#_\mu$ such that $\dim_H \Lambda^\# > 2 - \frac{1}{m}$. From Lemma 2 from [Du1] it follows that P_μ and $\Lambda^\#_\mu$ are homoclinically related. Therefore there exists a basic set Δ_μ such that $P_\mu \in \Delta_\mu$ and $\Lambda^\#_\mu \subset \Delta_\mu$. In particular, for $\mu \in \mathcal{R}_2 = \cap_{m\geq 1} A(m)$ we have $\dim_H H(P_\mu, f_\mu) = 2$. Set $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$. This proves (3.2).

Step 7. The last property (3.3) follows from the following Lemma

Lemma 21. *Let $\Lambda \subset M^2$ be a basic set of a surface diffeomorphism. Then*

$$\dim_H\{x \in \Lambda |\ \mathcal{O}^+(x) \text{ is dense in } \Lambda \text{ and } \mathcal{O}^-(x) \text{ is dense in } \Lambda\} = \dim_H \Lambda.$$

Theorem 2 is proved.

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